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On the normal modes of parallel flow of inviscid stratified fluid. Part 2. Unbounded flow with propagation at infinity

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The linear perturbations of the flow of a non-diffusive fluid are considered. The classification of the normal modes of parallel flow of an inviscid stratified fluid presented by Banks, Drazin & Zaturska (1976) is here extended to encompass modes which propagate at infinity. When the basic flow is unbounded and the buoyancy frequency is non-zero at infinity the five classes presented earlier are augmented by three further classes: for a given flow and wavenumber they are (*a*) a continuous class of non-singular stable modes which are modifications of internal gravity waves by shear; (*b*) a continuous class of stable modes which are singular at each critical layer but otherwise similar to those of class (*a*); and (*c*) a finite number of marginally stable singular modes with over-reflexion. This classification is illustrated by many new results. Some asymptotic properties of the stable and unstable modes are found for large values of the Richardson number and for long waves. Two prototype problems, in which the basic flows are a piecewise-linear shear layer and a triangular jet, are solved analytically. The modified internal gravity waves for a Bickley jet with uniform buoyancy frequency are treated to illustrate the complementary nature of the propagating and evanescent modes. This treatment is both analytical and numerical. The general ideas are further illustrated by a numerical study of the stability characteristics of a hyperbolic-tangent shear layer. Finally the modes for a basic flow of boundary-layer type are found in exact terms of a hypergeometric function.

1. Introduction

We have given an overall picture of the properties of the normal modes for a bounded shear flow of inviscid stratified fluid (Banks *et al.* 1976). These linear modes comprise Kelvin–Helmholtz instability, stable internal gravity waves, and other classes of stable waves. The picture included a classification of all waves, some general asymptotic results for small and large values of the Richardson number of the basic shear flow, and detailed numerical results for a few special flows chosen as exemplars. We may add here that this general picture is nicely complemented by the useful work of Yih (1974), Bell (1974) and Leibovich (1979). There have been so many papers on the mathematical problem, mostly giving detailed results for particular flows, and the problem itself has such a rich structure that a overview of its general properties is especially valuable.

Our previous results were applied and are applicable to unbounded flows for which the density tends to a constant, and therefore for which the buoyancy frequency tends

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to zero, at infinity. However, they are not applicable to modes which may propagate at infinity, and so are only partially applicable to the important class of unbounded basic flows in which the buoyancy frequency tends to a constant at infinity. Our present work gives a similar overall picture of the normal modes for these unbounded basic flows. The constant buoyancy frequency at infinity permits both stable modes representing the scattering of an incident wave by the basic shear flow and unstable modes which decay very slowly at infinity. At resonance between these two kinds of modes the amplitudes of the reflected and transmitted waves are infinitely greater than the amplitude of the incident waves; near resonance there is over-reflexion, the initially surprising (Jones 1968) but now well-known phenomenon whereby the amplitude of the reflected wave is much larger than that of the incident wave.

These problems are governed by the Taylor–Goldstein equation, namely

$$(U-c)(\phi''-\alpha^2\phi)-U''\phi+JN^2\phi/(U-c)=0, \quad \text{for} \quad -\infty<z<\infty, \tag{1}$$

where $U(z)$ is the dimensionless velocity of the basic flow in the horizontal x direction, $N(z)$ is the dimensionless local buoyancy frequency, sometimes called the Brunt–Väisälä frequency, J is the overall Richardson number, primes denote differentiations with respect to the dimensionless height z, and the stream function of the normal mode of disturbance is taken as $\phi(z)\exp\{i\alpha(x-ct)\}$. The complex eigenvalue c and eigenfunction ϕ are determined by the boundary conditions that

$$\phi \text{ is bounded as } z\to\pm\infty. \tag{2}$$

For given values of the wavenumber α one seeks the eigensolution (c, ϕ) and hence the stability characteristics of dynamically similar flows specified by J, $U(z)$ and $N(z)$. A given mode is stable if $\alpha c_i \leqslant 0$, and marginally stable if both $\alpha c_i = 0$ and there exist unstable modes at neighbouring values of the wavenumber and Richardson number.

As indicated above, we shall consider stably stratified flows for which there exist $\lim_{z\to\pm\infty} U(z), = U_{\pm\infty}$ say, and $\lim_{z\to\pm\infty} N(z), = N_{\pm\infty}$ say, so that wave propagation at infinity is possible. Therefore, in addition to the eigensolutions which we have already treated (Banks *et al.* 1976) and which satisfy the conditions that $\phi\to 0$ as $z\to\pm\infty$, there are also solutions which do not vanish at infinity. We call the former *bound states* and the latter *unbound states*, borrowing the names from quantum theory. As in quantum theory, it can easily be seen that any unbound state is linearly dependent on the solutions of scattering problems in which a given wave is incident upon the shear layer from one direction or the other and is then reflected and transmitted; indeed, these are the aspects of internal gravity waves considered by Booker & Bretherton (1967).

These ideas can be substantiated by noting that the solutions of the Taylor–Goldstein equation are exponential at infinity with exponents

$$\lambda_\pm = \{\alpha^2 - JN^2_{\pm\infty}/(U_{\pm\infty}-c)^2\}^{\frac{1}{2}}. \tag{3}$$

To be specific, we may define $\lambda_\pm$ to be the root for which $\operatorname{Re}\lambda_\pm > 0$ unless $\lambda_\pm$ is pure imaginary, in which case we use

$$\gamma_\pm = +\{JN^2_{\pm\infty}/(U_{\pm\infty}-c)^2-\alpha^2\}^{\frac{1}{2}} \tag{4}$$

instead. So the solutions of (1), (2) give $\phi \sim \text{const.}\exp(-\lambda_+ z)$ as $z \to +\infty$ unless λ_+ is pure imaginary, and $\phi \sim \text{const.}\exp(\lambda_- z)$ as $z \to -\infty$ unless λ_- is pure imaginary. Thus there is a bound state unless λ_+ or λ_- is pure imaginary.

To examine the unbound states, consider a wave of unit amplitude at $z = -\infty$, i.e. $\phi \sim \exp(\pm i\gamma_- z)$ as $z \to -\infty$. Then the phase velocity is given by

$$c = U_{-\infty} \pm \{JN^2_{-\infty}/(\alpha^2+\gamma^2_-)\}^{\frac{1}{2}}, \tag{5}$$

and the group velocity by

$$\mathbf{c}_g = \frac{\partial(\alpha c)}{\partial \alpha}\mathbf{i} + \frac{\partial(\alpha c)}{\partial \gamma_-}\mathbf{k}$$
$$= \frac{1}{(\alpha^2+\gamma^2_-)}\{(\alpha^2 U^2_{-\infty} + c\gamma^2_-)\mathbf{i} - \alpha\gamma_-(c-U_{-\infty})\mathbf{k}\}. \tag{6}$$

Therefore energy is propagated in the z-direction with velocity $-\alpha\gamma_-(c-U_{-\infty})/(\alpha^2+\gamma^2_-)$ and we require $\alpha\gamma_-(c-U_{-\infty}) < 0$ for propagation in the positive direction. We may take $\alpha \geqslant 0$ henceforth without loss of generality, and then require $c < U_{-\infty}$. This leads to the following boundary conditions for the scattering problem with an incident wave of unit amplitude at $z = -\infty$:

$$\phi \sim \exp(i\gamma_- z) + R\exp(-i\gamma_- z) \quad \text{as} \quad z \to -\infty, \tag{7}$$

and

$$\phi \sim \begin{Bmatrix} T\exp(i\gamma_+ z) & \text{if} \quad \gamma^2_+ > 0 \quad \text{and} \quad c < U_\infty \\ T\exp(-i\gamma_+ z) & \text{if} \quad \gamma^2_+ > 0 \quad \text{and} \quad c > U_\infty \\ V\exp(-\lambda_+ z) & \text{if} \quad \lambda^2_+ > 0 \end{Bmatrix} \quad \text{as} \quad z \to \infty, \tag{8}$$

where R, T and V are complex constants to be determined. We interpret R and T as reflexion and transmission coefficients respectively, and identify total reflexion if $\lambda^2_+ > 0$. If $c > U_{-\infty}$ we similarly take the incident wave such that $\phi \sim \exp(-i\gamma_- z)$ as $z \to -\infty$.

If $c > U_{\max}$ or $c < U_{\min}$ in this scattering problem, then ϕ is a complex solution of the real non-singular Taylor–Goldstein equation and its complex conjugate ϕ^* is an independent solution. Therefore their Wronskian $W(\phi, \phi^*) = \phi\, d\phi^*/dz - \phi^*\, d\phi/dz$ is a constant. Equating values of the Wronskian at $z = \pm\infty$ we get the usual result that

$$|R|^2 + \gamma_+|T|^2/\gamma_- = 1, \tag{9}$$

expressing the conservation of energy flux. Note that if $U_{\min} < c < U_{\max}$, then energy of the wave may be absorbed or fed by the mean flow at a critical layer, where $U(z) = c$, and the equation is singular so that (9) is invalid.

These ideas can be used to extend the classification of bound states by Banks *et al.* (1976). For fixed α^2, J, $U(z)$ and positive $N^2(z)$, each normal mode belongs to one of the following classes, any or many of which may be empty.

Bound states. These have eigenvalues c such that $\lambda^2_\pm$ is complex or positive and eigenfunctions which vanish at infinity.

(i) A finite number (possibly zero) of unstable modes with $\alpha c_i > 0$ and non-singular eigenfunctions. There certainly are none of these modes if $JN^2/U'^2 > \frac{1}{4}$ for all z (Miles 1961; Howard 1961).

(ii) An equal number of stable modes with $\alpha c_i < 0$ whose eigenvalues and eigenfunctions are complex conjugates of those of the previous class.

(iii) A finite number (almost always zero) of marginally stable modes for which $U_{\min} < c < U_{\max}$ and the eigenfunctions have branch points at the critical layers where $U(z) = c$. Miles (1961) called these singular neutral modes or SNM's.

(iv) A finite number or countable infinity of stable internal gravity waves modified by shear with $c > U_{\max}$ or $c < U_{\min}$ and non-singular eigenfunctions.

(v) A continuum of stable modes for which $U_{\min} < c < U_{\max}$ and the eigenfunctions have discontinuous derivatives at the critical layers. These are associated with algebraic rather than exponential decay of the disturbance.

Unbound states. These arise only for unbounded basic flows if $N^2 \not\to 0$ as $z \to \pm\infty$. They have eigenvalues such that $\gamma_+^2 > 0$ or $\gamma_-^2 < 0$ and eigenfunctions which are not integrable to infinity. They are specified by solution of the scattering problem.

(vi) A continuum of stable modes with $c > U_{\max}$ or $c < U_{\min}$ and non-singular eigenfunctions.

(vii) A continuum of stable modes with $U_{\min} < c < U_{\max}$ and eigenfunctions singular at the critical layers. These internal gravity waves may be partially absorbed or intensified by the basic flow at the critical layers (Booker & Bretherton 1967; Jones 1968).

(viii) A finite number (almost always zero) of marginally stable modes for which $U_{\min} < c < U_{\max}$ and the eigenfunctions have branch points at the critical layers. These describe infinite over-reflexion owing to resonance of an incident wave of class (vii) with the limit of an unstable mode of class (i), the singularity invalidating equation (9).

This classification can be verified with the help of the semicircle theorem (Howard 1961). In the complex c plane the eigenvalues of the unstable modes (i) lie inside the semicircle above (if $\alpha > 0$) its diameter $(U_{\min}, U_{\max})$ and the eigenvalues of the damped stable modes (ii) lie inside the semicircle below the diameter. The eigenvalues of the marginally stable modes (iii) and (viii) accordingly lie on the diameter itself, as do those of (v) and (viii). The eigenvalues of (iv) and (vi) lie on the real axis outside the diameter.

It is well-known, but nonetheless not always recognized, that a critical layer is a singularity of an individual stable normal mode not of a general disturbance of a basic flow. Such a disturbance, evolving in time, is represented by a superposition of singular modes in a Fourier–Laplace integral (cf. Eliassen, Höiland & Riis 1953). Although each stable mode may be singular at its own critical layer, different modes have different layers, and the integral smooths out all the layers so that the disturbance is non-singular everywhere at each instant. If, however, the flow is slightly unstable then only the wave components in a narrow band are unstable and so they will eventually dominate the disturbance. Thus a singular critical layer may develop in the limits as $t \to \infty$ and as $J \uparrow J_c$, where J_c is the value of J at marginal stability. This singularity will be removed by nonlinearity or diffusivity. For example, Brown & Stewartson (1978) have considered a problem of weak instability of a shear layer in an inviscid stratified fluid, in which a nonlinear critical layer develops.

2. The internal gravity waves for large J

In this section we shall solve some scattering problems asymptotically for large values of J. In this limit one might expect that buoyancy dominates inertia, so that reflexion and transmission are as if $U \equiv 0$. We shall see that this intuition needs qualification, in particular when c lies in the range of U, the analysis being rather different according to whether there is a critical layer or not.

(*a*) *No critical layer.* First suppose that $c^2/J \to$ const. as $J \to \infty$. In order to solve simply the first approximation to the solution of the scattering problem, let us further suppose that $N^2 =$ const, $= 1$ say, without loss of generality. For then we see that $R \to 0$ and $T \to 1$ as $J \to \infty$, and the solution is everywhere the incident wave

$$\phi \sim \phi_0 = \exp(i\gamma z) \quad \text{for fixed } z, \tag{10}$$

where $\gamma = +(J/c^2 - \alpha^2)^{\frac{1}{2}}$. (If $N^2(z) \neq$ const. then the analysis proceeds similarly but with more complications on taking ϕ_0 as the solution of the problem for $U \equiv 0$.)

This suggests that we try the expansions

$$\left.\begin{aligned} R = c^{-1}R_1 + c^{-2}R_2 + \ldots, \quad T = 1 + c^{-1}T_1 + c^{-2}T_2 + \ldots, \\ \phi = \phi_0 + c^{-1}\phi_1 + c^{-2}\phi_2 + \ldots \quad \text{as} \quad J \to \infty \quad \text{for fixed } \gamma. \end{aligned}\right\} \tag{11}$$

Now $\gamma_{\pm}^2 = \gamma^2 + 2c^{-3}JU_{\pm\infty} + O(c^{-4}J)$. Therefore

$$\left.\begin{aligned} &\phi_1 \sim ize^{i\gamma z}U_{-\infty}(\alpha^2+\gamma^2)/\gamma + R_1 e^{-i\gamma z} \quad \text{as} \quad z \to -\infty \\ \text{and} \quad &\phi_1 \sim \{T_1 + izU_\infty(\alpha^2+\gamma^2)/\gamma\}e^{i\gamma z} \quad \text{as} \quad z \to +\infty. \end{aligned}\right\} \tag{12}$$

Also the Taylor–Goldstein equation (1) may be rewritten exactly as

$$\phi'' + \gamma^2\phi = -c^{-1}\{U''/(1-U/c) + (\alpha^2+\gamma^2)(2U - U^2/c)/(1-U/c)^2\}\phi. \tag{13}$$

Equating powers of c^0 in this equation, we confirm the first approximation ϕ_0. Equating powers of c^{-1}, we find

$$\phi_1'' + \gamma^2\phi_1 = -\{U'' + 2(\alpha^2+\gamma^2)U\}\phi_0. \tag{14}$$

Multiplication by $e^{i\gamma z}$ and integration from $z = -\infty$ to ∞ gives

$$[e^{i\gamma z}(\phi_1' - i\gamma\phi_1)]_{-\infty}^{\infty} = -\int_{-\infty}^{\infty}\{U'' + 2(\alpha^2+\gamma^2)U\}e^{2i\gamma z}dz \tag{15}$$

and thence

$$R_1 = \frac{(\gamma^2-\alpha^2)}{\gamma}\left\{\frac{1}{2\gamma}(U_\infty - U_{-\infty}) - i\int_{-\infty}^{0}(U - U_{-\infty})e^{2i\gamma z}dz - i\int_0^\infty (U - U_\infty)e^{2i\gamma z}dz\right\}. \tag{16}$$

Similarly, multiplication of (14) by $e^{-i\gamma z}$ and integration from $z = -\infty$ to ∞ gives

$$T_1 = -\frac{(\alpha^2+\gamma^2)(U_\infty - U_{-\infty})}{2\gamma^2} + \frac{i(\alpha^2+\gamma^2)}{\gamma}\left\{\int_{-\infty}^{0}(U - U_{-\infty})dz + \int_0^\infty (U - U_\infty)dz\right\}. \tag{17}$$

It can be seen from (7), (8) and (10) that the limits as $J \to \infty$ and as $z \to \infty$ are not uniform and that therefore the unbounded terms arise in (12). None the less the results (16) and (17) can be more rigorously, but lengthily, derived by dividing out the exact exponential terms, then expanding the solution in inverse powers of c, and matching at the origin.

To illustrate these results, take $U = \tanh z$. Then it follows that

$$R \sim \pi(\gamma^2-\alpha^2)/c\gamma\sinh\pi\gamma, \quad T = 1 - (\alpha^2+\gamma^2)/c\gamma^2 + O(c^{-2}) \quad \text{as} \quad J \to \infty \quad \text{for fixed } \gamma.$$

Moreover, we can equate coefficients of c^{-2} in the energy flux relation (9) to deduce simply that $\operatorname{Re} T_2 = \frac{1}{2}(T_1^2 - R_1^2)$ and hence that

$$T_r = \operatorname{Re} T = 1 - \frac{\alpha^2+\gamma^2}{c\gamma^2} + \frac{1}{2c^2}\left[\left(\frac{\alpha^2+\gamma^2}{\gamma^2}\right)^2 - \left\{\frac{\pi(\gamma^2-\alpha^2)}{\gamma\sinh\pi\gamma}\right\}^2\right] + O(c^{-3}).$$

These asymptotic results agree well with direct integration of the Taylor–Goldstein equation. For example, taking $\alpha^2 = \frac{1}{2}$, $J = 15000$ and $c = 100$, we computed $T = 0{\cdot}985110 + 0{\cdot}00023i$ and $R = 0{\cdot}001359 - 0{\cdot}00067i$ directly, whereas the above one-term approximation for R gives $R = 0{\cdot}001360$ and the three-term approximation $T_r = 0{\cdot}985112$.

(*b*) *One critical layer*. Booker & Bretherton (1967) have shown how the reflexion of an incident internal gravity wave depends crucially upon the number and local structure of the critical layers. This leads to many cases in the asymptotic treatment. We shall take the simplest first, both for its own usefulness and as a prototype for other cases.

So we suppose that $U_{-\infty} < c < U_\infty$ and that $U(z_c) = c$ has a unique root z_c giving the height of the critical layer. We do not need to take N^2 constant here. Now the JWKB approximation applied to equation (1) gives

$$\phi(z) \sim \{U(z)-c\}^{\frac{1}{2}} \exp\left\{\pm iJ^{\frac{1}{2}} \int^z \frac{N(z_0)\,dz_0}{U(z_0)-c} + O(J^{-\frac{1}{2}})\right\} \quad \text{as} \quad J \to \infty \quad \text{for fixed } z \neq z_c. \tag{18}$$

Therefore we may take

$$\phi \sim \phi_1 = T'\left\{\frac{U(z)-c}{U_\infty - c}\right\}^{\frac{1}{2}} \exp\left\{iJ^{\frac{1}{2}} \int_b^z \frac{N(z_0)\,dz_0}{U(z_0)-c}\right\} \quad \text{for} \quad z > z_c, \tag{19}$$

and

$$\phi \sim \phi_2 = \left\{\frac{c-U(z)}{c-U_{-\infty}}\right\}^{\frac{1}{2}} \left[I' \exp\left\{iJ^{\frac{1}{2}} \int_z^a \frac{N(z_0)\,dz_0}{c-U(z_0)}\right\} + R' \exp\left\{-iJ^{\frac{1}{2}} \int_z^a \frac{N(z_0)\,dz_0}{c-U(z_0)}\right\}\right] \quad \text{for} \quad z < z_c, \tag{20}$$

choosing $a < z_c$ and $b > z_c$ arbitrarily. To be consistent with our normalization (7), (8) at infinity, we require

$$I' = \exp\left[-iJ^{\frac{1}{2}}\left\{\int_{-\infty}^a \frac{N(z_0)}{c-U(z_0)} - \frac{N_{-\infty}}{c-U_{-\infty}}\,dz_0 + \frac{aN_{-\infty}}{c-U_{-\infty}}\right\}\right], \quad R' = R/I'$$

and
$$T' = T\exp\left[-iJ^{\frac{1}{2}}\left\{\int_b^\infty \frac{N(z_0)}{U(z_0)-c} - \frac{N_\infty}{U_\infty - c}\,dz_0 - \frac{bN_\infty}{U_\infty - c}\right\}\right].$$

It will be helpful to re-arrange these solutions in the forms

$$\phi_1(z) = T'\left\{\frac{U(z)-c}{U_\infty - c}\right\}^{\frac{1}{2}} \exp\left\{iJ^{\frac{1}{2}} I_1(b,z) + \frac{iJ^{\frac{1}{2}} N_c}{U_c'} \log\left(\frac{z-z_c}{b-z_c}\right)\right\}, \tag{21}$$

and
$$\phi_2(z) = \left\{\frac{c-U(z)}{c-U_{-\infty}}\right\}^{\frac{1}{2}} \left[I' \exp\left\{iJ^{\frac{1}{2}} I_1(a,z) + \frac{iJ^{\frac{1}{2}} N_c}{U_c'} \log\left(\frac{z_c - z}{z_c - a}\right)\right\} + R' \exp\left\{-iJ^{\frac{1}{2}} I_1(a,z) - \frac{iJ^{\frac{1}{2}} N_c}{U_c'} \log\left(\frac{z_c-z}{z_c-a}\right)\right\}\right], \tag{22}$$

where we define

$$I_1(b,z) = \int_b^z \frac{N(z_0)}{U(z_0)-c} - \frac{N_c}{U_c'(z_0-z_c)}\,dz_0.$$

The JWKB approximation breaks down near $z = z_c$, so we consider an inner limit as $z \to z_c$ for fixed J. In this limit the solution of the Taylor–Goldstein equation (1) is

$$\phi \sim \phi_i = A(z-z_c)^{\frac{1}{2}+i\mu} + B(z-z_c)^{\frac{1}{2}-i\mu} \quad \text{as} \quad z \downarrow z_c, \tag{23}$$

where $\mu = +(JN_c^2/U_c'^2 - \frac{1}{4})^{\frac{1}{2}}$ and A and B are arbitrary constants. In accord with Booker & Bretherton's generalization of Tollmien's result (Lin 1955) for interpreting the singularity at a critical layer in an inviscid fluid by taking the limit as $c_i \downarrow 0$ (and noting that our hypotheses imply that $U_c' > 0$), we deduce that

$$\phi_i = -ie^{\pi\mu} A(z_c - z)^{\frac{1}{2}+i\mu} - ie^{-\pi\mu} B(z_c - z)^{\frac{1}{2}-i\mu} \quad \text{as} \quad z \uparrow z_c. \tag{24}$$

We can now match (21) with (23), taking $\lim_{z\uparrow\infty} \phi_i \sim \lim_{z\downarrow z_0} \phi_1$ as $J \to \infty$ and therefore

$$A \sim T' \left(\frac{U_c'}{U_\infty - c}\right)^{\frac{1}{2}} \exp\left\{iJ^{\frac{1}{2}} I_1(b, z_c) - \frac{iJ^{\frac{1}{2}} N_c}{U_c'} \log(b - z_c)\right\} \quad \text{and} \quad B \to 0 \quad \text{as} \quad J \to \infty.$$

Similarly, matching (22) with (24), we find

$$-ie^{\pi\mu} A \sim I' \left(\frac{U_c'}{c - U_{-\infty}}\right)^{\frac{1}{2}} \exp\left\{iJ^{\frac{1}{2}} I_1(a, z_c) - \frac{iJ^{\frac{1}{2}} N_c}{U_c'} \log(z_c - a)\right\}$$

$$\text{and} \quad R' \to 0 \quad \text{as} \quad J \to \infty.$$

It follows that

$$T \sim \exp\left[iJ^{\frac{1}{2}}\left\{\int_b^\infty \frac{N(z_0)}{U(z_0) - c} - \frac{N_\infty}{U_\infty - c} dz_0 - \frac{bN_\infty}{U_\infty - c} - \int_{-\infty}^a \frac{N(z_0)}{c - U(z_0)} - \frac{N_{-\infty}}{c - U_{-\infty}} dz_0 - \frac{aN_{-\infty}}{c - U_{-\infty}}\right\}\right]$$

$$\times\, ie^{-\pi\mu} \left(\frac{U_\infty - c}{c - U_{-\infty}}\right)^{\frac{1}{2}} \left(\frac{b - z_c}{z_c - a}\right)^{i\mu} \exp\{iJ^{\frac{1}{2}} I_1(a, b)\} \tag{25}$$

and

$$R \to 0 \quad \text{as} \quad J \to \infty. \tag{26}$$

The basic flow absorbs almost all of the energy of the incident wave at the critical layer, as first shown by Booker & Bretherton (1967), so the conservation law (9) is invalid. We see that T is exponentially small and R is small at an as yet unspecified order.

To illustrate the result (25), we consider the profile

$$U = \left\{\begin{array}{lll} 1 & \text{for} & z > 1 \\ z & \text{for} & -1 \leqslant z \leqslant 1 \\ -1 & \text{for} & z < -1 \end{array}\right\} \quad \text{and} \quad N^2 = 1. \tag{27}$$

Choosing $b = 1$ and $a = -1$, we find $z_c = c$ and $I_1(a, b) = 0$, and therefore

$$T \sim ie^{-\pi\mu} \left(\frac{1-c}{1+c}\right)^{i\mu+\frac{1}{2}} \exp\{-2iJ^{\frac{1}{2}} c/(1-c^2)\} \quad \text{as} \quad J \to \infty, \tag{28}$$

in agreement with the result (63) obtained by solution of the problem in explicit terms of Bessel functions.

(*c*) *Two critical layers*. The principles applied in the previous subsection may be applied to other cases, but the technical difficulties of the method increase with the number of critical layers. So we shall present just one more case, namely transmission through a symmetric jet for which $U_{\pm\infty} = 0$, $U_{\max} = U(0) = 1$, $0 < c < 1$ and $U(z) = c$ has only two roots, say $\pm z_c$.

Then equation (1) and the radiation conditions (7), (8) at $z = \pm\infty$ may be satisfied to the JWKB approximation by

$$\phi \sim \phi_1 = T' \left\{\frac{c - U(z)}{c}\right\}^{\frac{1}{2}} \exp\left\{-iJ^{\frac{1}{2}} \int_b^z \frac{N(z_0)\, dz_0}{c - U(z_0)}\right\} \quad \text{for} \quad z_c < z, \tag{29}$$

$$\phi \sim \phi_2 = \{U(z)-c\}^{\frac{1}{2}}\left[A\exp\left\{iJ^{\frac{1}{2}}\int_0^z \frac{N(z_0)\,dz_0}{U(z_0)-c}\right\}+B\exp\left\{-iJ^{\frac{1}{2}}\int_0^z \frac{N(z_0)\,dz_0}{U(z_0)-c}\right\}\right]$$
$$\text{for}\quad -z_c < z < z_c, \tag{30}$$

and

$$\phi \sim \phi_3 = \left\{\frac{c-U(z)}{c}\right\}^{\frac{1}{2}}\left[I'\exp\left\{iJ^{\frac{1}{2}}\int_z^{-b} \frac{N(z_0)\,dz_0}{c-U(z_0)}\right\}+R'\exp\left\{-iJ^{\frac{1}{2}}\int_z^{-b} \frac{N(z_0)\,dz_0}{c-U(z_0)}\right\}\right]$$
$$\text{for}\quad z < -z_c, \tag{31}$$

as $J \to \infty$, where A, B, R' and T' are complex constants to be determined, and b may be chosen to have any value greater than z_c. To be consistent with the normalization of (7), (8) we require

$$I' = \exp\left[-\frac{iJ^{\frac{1}{2}}}{c}\left\{-bN_{-\infty}+\int_{-\infty}^{-b}\left(\frac{cN(z_0)}{c-U(z_0)}-N_{-\infty}\right)dz_0\right\}\right],\quad R' = R/I',$$

and $$T' = T\exp\left[-\frac{iJ^{\frac{1}{2}}}{c}\left\{bN_{\infty}-\int_{b}^{\infty}\left(\frac{cN(z_0)}{c-U(z_0)}-N_{\infty}\right)dz_0\right\}\right].$$

The inner solutions near the two critical layers at $z = \pm z_c$ have the same form as in the previous subsection. The matching is similar to, but more intricate than that before. With the further assumption of a symmetric form for $N(z)$, namely, $N(z) = N(-z)$, it eventually gives

$$T \sim e^{-2\mu\pi}\left(\frac{b-z_c}{z_c}\right)^{2iJ^{\frac{1}{2}}N_c/U_c'} I'\exp\left[iJ^{\frac{1}{2}}\left\{-2I_2(0,b)+\frac{bN_\infty}{c}-\int_b^\infty\left(\frac{N(z_0)}{c-U(z_0)}-\frac{N_\infty}{c}\right)dz_0\right\}\right], \tag{32}$$

$$R \to 0 \quad \text{as} \quad J \to \infty, \tag{33}$$

where $$I_2(b,z) = \int_b^z \left(\frac{N(z_0)}{c-U(z_0)}+\frac{N_c}{U_c'(z_0-z_c)}\right)dz_0,\quad U_c' = U'(z_c)$$

and $N_c = N(z_c)$. Note that $\mu = +(JN_c^2/U_c'^2-\frac{1}{4})^{\frac{1}{2}} \sim J^{\frac{1}{2}}N_c/|U_c'|$ as $J \to \infty$, and that $U'(-z_c) = -U_c' > 0$, and also that b appears in the expression for T in such a way that the solution is independent of the choice of b.

For the example of the triangular jet, namely

$$U = \begin{cases} 0 & \text{for}\quad |z| > 1 \\ 1-|z| & \text{for}\quad |z| < 1 \end{cases} \quad \text{and}\quad N^2 \equiv 1, \tag{34}$$

we may choose $b = 1$, find $z_c = 1-c$, and then use (32) to deduce that

$$T \sim e^{-2\mu\pi+2iJ^{\frac{1}{2}}/c}\left(\frac{1-c}{c}\right)^{2iJ^{\frac{1}{2}}} \quad \text{as}\quad J \to \infty, \tag{35}$$

in agreement with the result in (69) obtained by direct solution of the problem.

(*d*) *An integral equation to deduce reflexion coefficients*. It appears that the matching must be taken further to give the leading asymptotic behaviour of R as $J \to \infty$. However, we may estimate R more simply by use of an integral equation. A jet is the simplest type of flow to be treated in this way. So, to delineate the method, let us first suppose that $U_{-\infty} = U_\infty$ and $N_{-\infty} = N_\infty \neq 0$. Then we may make a Galilean transformation so that $U_\infty = 0$ and choose the scale of J so that $N_\infty = 1$. Also we may suppose that $c > 0$ without loss of generality.

Following the well-known method described by Morse & Feshbach (1953, p. 1071) we first rewrite the Taylor–Goldstein equation exactly in the form (13), or

$$\phi'' + \gamma^2\phi = S\phi, \tag{36}$$

where
$$S(z) = \frac{J}{c^2}\left\{1 - \left(1 - \frac{U}{c}\right)^{-2}\right\} + \frac{U''}{U-c}. \tag{37}$$

The radiation conditions (7), (8) for the jet give

$$\phi \sim \begin{Bmatrix} T\,e^{-i\gamma z} & \text{as} & z \to \infty, \\ e^{-i\gamma z} + R\,e^{i\gamma z} & \text{as} & z \to -\infty. \end{Bmatrix} \tag{38}$$

Then, using the Green's function

$$G(z, z_0) = -\frac{i}{2\gamma}\exp\{-i\gamma\,|z - z_0|\}, \tag{39}$$

we deduce the exact integral equation

$$\phi(z_0) = e^{-i\gamma z_0} - \int_{-\infty}^{\infty} G(z, z_0)\,S(z)\,\phi(z)\,dz. \tag{40}$$

Taking the limit as $z_0 \to -\infty$ in this equation, and then isolating the coefficient of $e^{i\gamma z_0}$, we find

$$R = \frac{i}{2\gamma}\int_{-\infty}^{\infty} e^{-i\gamma z}\,S(z)\,\phi(z)\,dz. \tag{41}$$

We may now find R asymptotically by use of the JWKB approximation to ϕ in the exact expression (41). If $c > U_{\max}$, then there is no critical layer, and $\phi \sim e^{-i\gamma z}$ as $J \to \infty$ for fixed γ. Therefore (41) gives

$$R \sim -\frac{i}{2\gamma c}\int_{-\infty}^{\infty}\{U'' + 2(\alpha^2 + \gamma^2)U\}\,e^{-2i\gamma z}\,dz \quad \text{as} \quad J \to \infty, \tag{42}$$

in agreement with (15), the signs of γ being different because here $c > U_{-\infty}$ (see the discussion following (8)). Note that $R = O(J^{-\frac{1}{2}})$ as $J \to \infty$.

If the jet is symmetric with two critical layers at $z = \pm z_c$ then we can use the JWKB approximation (29)–(31) to ϕ, ϕ_0 say, as $J \to \infty$ for fixed c. Now ϕ_0 is exponentially small for $z > -z_c$, and $R \to 0$ as $J \to \infty$. Therefore we may replace the upper bound infinity of the integral (41) by $-z_c(1-\epsilon)$ where $0 < \epsilon \ll 1$ so that

$$R \sim \frac{iI'}{2\gamma}\int_{-\infty}^{-z_c(1-\epsilon)} e^{-i\gamma z}\left[\frac{J}{c^2}\left\{1 - \left(1 - \frac{U}{c}\right)^{-2}\right\} + \frac{U''}{U-c}\right]\phi_0\,dz \quad \text{as} \quad J \to \infty, \tag{43}$$

where ϕ_0 is now replaced by only its incident wave component. The integrand is singular at $z = -z_c$, but integration through the critical layer gives a finite value for R.

As an illustration we consider the triangular jet (34). Then

$$U'' = \delta(z+1) - 2\delta(z) + \delta(z-1),$$

and $z_c = 1 - c$. At length (43) gives

$$R \sim -\frac{i e^{2iJ^{\frac{1}{2}}/c}}{4J^{\frac{1}{2}}} \quad \text{as} \quad J \to \infty \tag{44}$$

in agreement with the result (69) obtained by direct solution of the problem.

It may be noted in summary that in subsection (*a*) we have taken the limit as $J \to \infty$ for fixed J/c^2, α, $U(z)$ and $N \equiv 1$, in subsections (*b*)–(*d*) as $J \to \infty$ for fixed c, α, $U(z)$ and $N(z)$. In the next section we shall take the limit as $\alpha \to 0$ for fixed J/α^2 and find c for the unstable modes. These asymptotic results complement those of Grimshaw (1976*a*) in what is essentially the limit as $\alpha \to \infty$ for fixed J/α^2, c and analytic $N(z)$ and $U(z)$. He deduced corresponding results when the basic shear flow contains a critical layer and when it does not by a powerful application of the JWKB method and the theory of a function of a complex variable. Among many interesting results, he showed that $R \to 0$ exponentially in his limit, as may also be seen from (41) in our limit if U is analytic. It is, however, shown in (44) that $R \to 0$ algebraically for a profile U which is not analytic.

3. Unstable long waves for basic shear layers

We shall now examine some properties of the unstable modes which are associated with over-reflexion and resonance in the scattering problem. Silcock (1975) has examined these properties, both asymptotically and numerically, for several jets, finding for each jet a mode of instability which tends to a propagating wave as its stability boundary is approached. Thus he found marginally stable modes as limits of unstable bound states, which are effectively modes behaving like (7) and (8) at infinity with infinite R and T. This gives a resonance in the scattering problem, which we shall exemplify in § 7.

Because of the comprehensiveness of Silcock's (1975) treatment of the instability of jets, we shall confine our treatment to shear layers. Now Alterman (1961) considered the simplest shear layer, namely the vortex sheet

$$U = \begin{Bmatrix} 1 & \text{for} & z > 0 \\ -1 & \text{for} & z < 0 \end{Bmatrix} \quad \text{and} \quad N^2 = 1, \tag{45}$$

and found modes with

$$c^2 = J/2\alpha^2 - 1. \tag{46}$$

Drazin & Howard (1966, pp. 46–57) further found a stable propagating mode and noted that the stability characteristics for the vortex sheet should be the same as those of long waves for *any* smoothly varying shear layer with $U_{\pm\infty} = \pm 1$ and $N_{\pm\infty} = 1$. This propagating mode is now recognizable as a resonating mode with R, $T = \infty$, and will be shown to be the vestige (in the limit of infinite wavelength) of the stability boundary of an unstable propagating mode of a rapidly varying smooth shear layer.

Thus the long-wave theory appears as a key to the problem. So we shall quote the eigenvalue relation (Drazin & Howard 1966, equation (5.29), after correction of misprints) for bound states as $\alpha \to 0$ for fixed J/α^2 and $N^2 = 1$:

$$\begin{aligned} 0 = \lambda_+ W_\infty^2 + \lambda_- W_{-\infty}^2 + \alpha^2 \left[\int_0^\infty W^2 - W_\infty^2 \, dz + \int_{-\infty}^0 W^2 - W_{-\infty}^2 dz \right] \\ - \lambda_+ \lambda_- \left[W_{-\infty}^2 \int_0^\infty 1 - W_\infty^2 / W^2 \, dz + W_\infty^2 \int_{-\infty}^0 1 - W_{-\infty}^2 / W^2 \, dz \right] + \ldots, \end{aligned} \tag{47}$$

where $W(z) = U(z) - c$ and $W_{\pm\infty} = U_{\pm\infty} - c$.

For a general shear layer with $U_{\pm\infty} = \pm 1$, we find $c = 0$ when

$$J/\alpha^2 = 1-\left(\tfrac{1}{2}\alpha\int_{-\infty}^{\infty} 1-U^2\,dz\right)^2+o(\alpha^2) \quad \text{as} \quad \alpha \to 0. \tag{48}$$

It is also readily shown from (47) that for the general antisymmetric shear layer with $U_{\pm\infty} = \pm 1$ and $U(-z) = -U(z)$, and for $1 < J/\alpha^2 < 2$, there is an unstable mode such that

$$c \sim \frac{-i\alpha(J/\alpha^2-1)^{\frac{1}{2}}\left\{\int_{-\infty}^{\infty}(U^2-1)\,dz-(J/\alpha^2-1)\int_{-\infty}^{\infty}(1-U^{-2})\,dz\right\}}{2(2-J/\alpha^2)}, \tag{49}$$

as $\alpha \to 0$ for fixed J/α^2.

For the piecewise-linear shear layer (27), (47) gives

$$\begin{aligned}0 = (1-c)^2\{1-J/\alpha^2(1-c)^2\}^{\frac{1}{2}}+(1+c)^2\{1-J/\alpha^2(1+c)^2\}^{\frac{1}{2}}\\ -4\alpha[\tfrac{1}{3}+\{1-J/\alpha^2(1-c)^2\}^{\frac{1}{2}}\{1-J/\alpha^2(1+c)^2\}^{\frac{1}{2}}]+\ldots\end{aligned} \tag{50}$$

as $\alpha \to 0$ for fixed J/α^2; hence, or from (48), $c = 0$ when

$$J/\alpha^2 = 1-\tfrac{4}{9}\alpha^2+o(\alpha^2) \quad \text{as} \quad \alpha \to 0. \tag{51}$$

For $1 < J/\alpha^2 < 2$, (50) (or indeed (49)) gives an unstable mode with

$$c \sim \frac{2i\alpha(J/\alpha^2-\frac{2}{3})(J/\alpha^2-1)^{\frac{1}{2}}}{(2-J/\alpha^2)} \quad \text{as} \quad \alpha \to 0. \tag{52}$$

The formula (52) clearly fails for J/α^2 near 2; this case is dealt with by noting that for $c \neq 0$ (50) becomes

$$2c(1+c^2)-(J/\alpha^2)\,c = 8\alpha i(1-c^2)/3+\ldots$$

as $\alpha \to 0$. When both α and $2-J/\alpha^2$ are small, so that c too is small, this is approximated by

$$c^3+(1-\tfrac{1}{2}J/\alpha^2)\,c-4\alpha i/3 = 0.$$

This gives two admissible unstable roots c and one inadmissible stable root with $c = ic_i$ and $\alpha c_i < 0$. The two unstable roots are pure imaginary when

$$\tfrac{1}{2}J/\alpha^2 < 1-(12\alpha^2)^{\frac{1}{3}},$$

but are complex conjugates when $\frac{1}{2}J/\alpha^2 > 1-(12\alpha^2)^{\frac{1}{3}}$. The critical value

$$\tfrac{1}{2}J/\alpha^2 = 1-(12\alpha^2)^{\frac{1}{3}}+\ldots$$

as $\alpha \to 0$ *is* thus strongly suggestive of bifurcation: this has a parallel in the study in the following paragraph which will be shown to be consistent with the numerical results of § 7.

For the basic shear layer with $U = \tanh z$, we find from (48) that $c = 0$ when

$$J/\alpha^2 = 1-\alpha^2+o(\alpha^2) \quad \text{as} \quad \alpha \to 0.$$

For $1 < J/\alpha^2 < 2$, (49) gives an unstable mode with

$$c \sim \frac{i\alpha(J/\alpha^2)(J/\alpha^2-1)^{\frac{1}{2}}}{(2-J/\alpha^2)} \tag{53}$$

as $\alpha \to 0$ for fixed J/α^2. We note that this result is identical with equation (22) of Blumen, Drazin & Billings (1975) when J/α^2 is replaced by the square of the Mach number. It follows from (47) that

$$8c(1+c^2) - 4cJ/\alpha^2 = 4\alpha i(1-c^2)\left\{2 + c\log\left(-\frac{1+c}{1-c}\right)\right\} + \dots \quad \text{as} \quad \alpha \to 0, \tag{54}$$

and therefore

$$c^2 = \tfrac{1}{2}J/\alpha^2 - 1 + \frac{\alpha i(1-c^2)}{c}\left\{1 + \tfrac{1}{2}c\log\left(-\frac{1+c}{1-c}\right)\right\} + o(\alpha) \quad \text{as} \quad \alpha \to 0.$$

If $J/\alpha^2 < 2$, then this gives

$$c^2 = -(1-\tfrac{1}{2}J/\alpha^2) + \frac{\alpha(2-J/\alpha^2)}{(1-\frac{1}{2}J/\alpha^2)^{\frac{1}{2}}}\{1 + (1-\tfrac{1}{2}J/\alpha^2)\tan^{-1}(1-\tfrac{1}{2}J/\alpha^2)^{-\frac{1}{2}}\} + \dots \tag{55}$$

as $\alpha \to 0$ for fixed J/α^2. If $2 < J/\alpha^2 < 4$, then

$$\left.\begin{aligned} c_r &= \pm(\tfrac{1}{2}J/\alpha^2 - 1)^{\frac{1}{2}}\{1 + \tfrac{1}{4}\pi\alpha(2-\tfrac{1}{2}J/\alpha^2)/(\tfrac{1}{2}J/\alpha^2-1) + \dots\}, \\ c_i &= \frac{\frac{1}{2}\alpha(2-\frac{1}{2}J/\alpha^2)}{(\frac{1}{2}J/\alpha^2-1)}\left\{1 + (\tfrac{1}{2}J/\alpha^2-1)^{\frac{1}{2}}\log\left[\frac{1+(\frac{1}{2}J/\alpha^2-1)^{\frac{1}{2}}}{(2-\frac{1}{2}J/\alpha^2)^{\frac{1}{2}}}\right] + \dots\right\} \end{aligned}\right\} \tag{56}$$

as $\alpha \to 0$ for fixed J/α^2. If $J/\alpha^2 > 4$, then there is no instability as $\alpha \to 0$. A stability boundary thus approaches the curve $J/\alpha^2 = 4$ as $\alpha \to 0$; this limit is an intricate one because $c \to \pm 1$ and the critical layer recedes to infinity. We shall exemplify this numerically in § 7. If, however, both α and $2 - J/\alpha^2$ are small so that c too is small, (54) is approximated by

$$c^3 + (1-\tfrac{1}{2}J/\alpha^2)c - i\alpha = 0. \tag{57}$$

Analysis similar to that for the case of the piecewise-linear shear layer indicates bifurcation along the curve

$$J/\alpha^2 = 2 - 3(2\alpha^2)^{\frac{1}{3}} + \dots \quad \text{as} \quad \alpha \to 0. \tag{58}$$

It can be seen that for the hyperbolic-tangent shear layer the stability boundary meets the (J/α^2)-axis when the critical layer recedes to infinity, because $c \to U_{\pm\infty} = \pm 1$ as $\alpha \to 0$ along the boundary. Now if $\alpha = 0$ we get the eigenvalue relation (46) for the vortex sheet and if further $c = \pm 1$ we deduce $J/\alpha^2 = 4$. Thus we expect that the boundary meets the axis at $J/\alpha^2 = 4$ for *all* shear layers (with $U_\infty = 1$ and $U_{-\infty} = -1$) but that the *way* the boundary meets the axis depends upon the nature of the critical layer and hence the way $U(z) \to \pm 1$ as $z \to \pm\infty$. This behaviour is similar to that for a shear layer in a compressible uniform perfect gas (Blumen *et al.* 1975), but for that problem $c \to \pm 1$ as the Mach number tends to infinity.

4. The piecewise-linear mixing layer between two uniform streams

For the shear layer (27), we can solve the Taylor–Goldstein equation (1) piecewise, join up the solutions at $z = \pm 1$ by use of the continuity of $(U-c)\phi' - U'\phi$ and ϕ (which follows from the continuity of the pressure and normal velocity of the fluid at the disturbed interface), and satisfy the conditions (7) and (8) at infinity.

If there is a critical layer, then $-1 < c < 1$ and there is only one layer. Thus $z_c = c$ and we may satisfy (1), (7) and (8) by taking

$$\phi = \left\{\begin{array}{lr} T e^{i\gamma_+ z} & \text{for} \quad 1 < z \\ {[AI_\nu\{\alpha(z-c)\} + BI_{-\nu}\{\alpha(z-c)\}](z-c)^{\frac{1}{2}}} & \text{for} \quad -1 < z < 1 \\ e^{-i\gamma_- z} + R e^{i\gamma_- z} & \text{for} \quad z < -1, \end{array}\right\} \tag{59}$$

where $\gamma_\pm = +\{J/(\pm 1-c)^2-\alpha^2\}^{\frac{1}{2}}$, $\nu = +(\frac{1}{4}-J)^{\frac{1}{2}}$ and I_ν is a modified Bessel function. We must interpret the branch point at $z = c$ in the usual limit as $c_i \downarrow 0$ so that $\log(z-c) = \log(c-z)-\pi i$ for $z < c$.

The boundary conditions at $z = \pm 1$, after some re-arrangement, give

$$(1-c)^{\frac{1}{2}}\{AI_\nu(k_+)+BI_{-\nu}(k_+)\}-T\,e^{i\gamma_+} = 0,$$
$$F_+(k_+)\,A+F_-(k_+)\,B = 0,$$
$$(-1-c)^{\frac{1}{2}}\{AI_\nu(k_-)+(BI_{-\nu}(k_-)\}-R\,e^{-i\gamma_-} = e^{i\gamma_-},$$
$$G_+(k_-)\,A+G_-(k_-)^{\frac{1}{2}}B = -2i\gamma_-(-1-c)^{\frac{1}{2}}\,e^{i\gamma_-},$$

where $k_\pm = \alpha(\pm 1-c)$, $F_\pm(x) = xI'_{\pm\nu}(x)-\{\frac{1}{2}+i\gamma_+(1-c)\}\,I_{\pm\nu}(x)$ and

$$G_\pm(x) = xI'_{\pm\nu}(x)-\{\tfrac{1}{2}-i\gamma_-(1+c)\}\,I_{\pm\nu}(x).$$

If $J > \frac{1}{4}$,† we replace ν by $i\mu$ where $\mu = +(J-\frac{1}{4})^{\frac{1}{2}}$. Therefore

$$T = -4i\gamma_-\sin\pi\nu\exp\{i(\gamma_--\gamma_+)\}/\pi\Delta \tag{60}$$

and
$$R = e^{2i\gamma_-}\left[-1+\frac{2i\gamma_-}{\Delta}\left(-\frac{1+c}{1-c}\right)^{\frac{1}{2}}\{I_\nu(k_-)\,F_-(k_+)-I_{-\nu}(k_-)\,F_+(k_+)\}\right], \tag{61}$$

where the discriminant of the homogeneous system is essentially

$$\Delta = \{F_+(k_+)\,G_-(k_-)-F_-(k_+)\,G_+(k_-)\}/(1-c)^{\frac{1}{2}}(-1-c)^{\frac{1}{2}}. \tag{62}$$

It follows at length that

$$\Delta = -2J^{\frac{1}{2}}\,e^{2\pi\mu}\left(\frac{1+c}{1-c}\right)^{i\mu}\{1+O(J^{-\frac{1}{2}})\}/\pi(1-c)^{\frac{1}{2}}(-1-c)^{\frac{1}{2}} \quad \text{as} \quad J\to\infty$$

and thence that

$$T \sim ie^{-\pi\mu+i(\gamma_--\gamma_+)}\left(\frac{1-c}{1+c}\right)^{i\mu+\frac{1}{2}} \quad \text{and} \quad R \sim -e^{2i\gamma_-}/4J^{\frac{1}{2}} \quad \text{as} \quad J\to\infty. \tag{63}$$

Note that $\mu \sim J^{\frac{1}{2}}$ and $\gamma_\pm \sim J^{\frac{1}{2}}/(1\mp c)$ as $J\to\infty$.

For the bound states, $\phi\to 0$ as $z\to\pm\infty$. Thus effectively $R, T = \infty$ or

$$\Delta = 0. \tag{64}$$

This is the eigenvalue relation for the bound states. It gives

$$J/\alpha^2 = 2(1+c^2)-8i\alpha(1-c^2)/3c+\ldots \quad \text{as} \quad \alpha\to 0,$$

in agreement with (50), etc.

5. The triangular jet

For the triangular jet (34) we can similarly solve the problem explicitly. Again let us suppose first that $0 < c < 1$ so that there are two critical layers at $\pm z_c$, where $z_c = 1-c$. We now satisfy (1), (7) and (8) by taking

$$\phi = \begin{cases} Te^{-i\gamma z} & \text{for} \quad z > 1, \\ (z-1+c)^{\frac{1}{2}}\,[A_+I_\nu\{\alpha(z-1+c)\}+B_+I_{-\nu}\{\alpha(z-1+c)\}] & \text{for} \quad 0 < z < 1, \\ (z+1-c)^{\frac{1}{2}}\,[A_-I_\nu\{\alpha(z+1-c)\}+B_-I_{-\nu}\{\alpha(z+1-c)\}] & \text{for} \quad -1 < z < 0, \\ e^{-i\gamma z}+R\,e^{i\gamma z} & \text{for} \quad z < -1. \end{cases} \tag{65}$$

† Although the profile is monotonic, theorem (iv) of Miles (1963), excluding the occurrence of singular neutral modes for $J > \frac{1}{4}$, applies only to bound states.

The boundary conditions of continuity of $(U-c)\phi'-U'\phi$ and ϕ at $z=0, \pm 1$, after some re-arrangement, give

$$c^{\frac{1}{2}}\{A_+I_\nu(k_1)+B_+I_{-\nu}(k_1)\}-Te^{i\gamma}=0,$$
$$F_+(k_1)A_++F_-(k_1)B_+=0,$$
$$(-1+c)^{\frac{1}{2}}\{A_+I_\nu(-k_0)+B_+I_{-\nu}(-k_0)\}-(1-c)^{\frac{1}{2}}\{A_-I_\nu(k_0)+B_-I_{-\nu}(k_0)\}=0,$$
$$(-1+c)^{\frac{1}{2}}\{A_+L_+(-k_0)+B_+L_-(-k_0)\}+(1-c)^{\frac{1}{2}}\{A_-L_+(k_0)+B_-L_-(k_0)\}=0,$$
$$(-c)^{\frac{1}{2}}\{A_-I_\nu(-k_1)+B_-I_{-\nu}(-k_1)\}-Re^{-i\gamma}=e^{i\gamma},$$
$$A_-F_+(-k_1)+B_-F_-(-k_1)=-2i\gamma(-c)^{\frac{1}{2}}e^{i\gamma},$$

where $\quad k_0=\alpha(1-c), \quad k_1=\alpha c, \quad F_\pm(x)=xI'_{\pm\nu}(x)-(\frac{1}{2}-i\nu c)I_{\pm\nu}(x)$

and $\quad L_\pm(x)=xI'_{\pm\nu}(x)-\frac{1}{2}I_{\pm\nu}(x).$

These six simultaneous equations have solution

$$\Delta T=8\gamma c(1-c)\sin^2\nu\pi\,.\,e^{2i\gamma}/\pi^2 \tag{66}$$

and
$$\Delta R=-2i(1-c)\,e^{2i\gamma}\{e^{-i\nu\pi}F_+(k_1)L_-(k_0)-e^{i\nu\pi}F_-(k_1)L_+(k_0)\} \times\{e^{-i\nu\pi}G_+(k_1)I_{-\nu}(k_0)-e^{i\nu\pi}G_-(k_1)I_\nu(k_0)\}, \tag{67}$$

where the discriminant of the homogeneous system is essentially

$$\Delta=2i(1-c)\{e^{-i\nu\pi}F_+(k_1)L_-(k_0)-e^{i\nu\pi}F_-(k_1)L_+(k_0)\}\{e^{-i\nu\pi}F_+(k_1)I_{-\nu}(k_0) -e^{i\nu\pi}F_-(k_1)I_\nu(k_0)\}. \tag{68}$$

In obtaining these results care must be taken to interpret correctly $(-1)^{\frac{1}{2}}$: this depends on whether the term originates from considerations in the upper or lower part of the profile.

It follows at length that

$$\Delta\sim-2ce^{2i\mu}(1-c)^{1-2i\mu}J^{\frac{1}{2}}e^{4\pi\mu}/\pi^2 \quad \text{as} \quad J\to\infty$$

and thence that

$$T\sim\left(\frac{1-c}{c}\right)^{2i\mu}e^{2i\gamma-2\mu\pi} \quad \text{and} \quad R\sim-ie^{2i\gamma}/4J^{\frac{1}{2}} \quad \text{as} \quad J\to\infty. \tag{69}$$

Note that $\mu\sim J^{\frac{1}{2}}$ and $\gamma\sim J^{\frac{1}{2}}/c$ as $J\to\infty$.

For the bound states, we have $\Delta=0$. This has been solved numerically by Silcock (1975).

6. The Bickley jet

For the Bickley jet,

$$U=\operatorname{sech}^2 z \quad \text{and} \quad N^2=1, \tag{70}$$

the unstable modes have been investigated numerically by Silcock (1975), who found their complicated structure with just one varicose (odd ϕ) and more than one sinuous (even ϕ) unstable mode. Also some modified internal gravity waves exist as bound states, for which

$$c=1+2J/n(n+2)+o(J) \quad \text{as} \quad J\downarrow 0 \quad \text{for} \quad n=1,2,\ldots, \tag{71}$$

in accord with Banks *et al.* (1976, equation (43)). It is instructive to see what happens to these bound states as J increases, for they illustrate the complementary nature of the bound and unbound states according as $\gamma^2 = \alpha^2 - J/c^2 < 0$ or $\gamma^2 > 0$ respectively. The bound states exist only if $J < c^2\alpha^2$, and we shall show that the infinity of them given by (71) for infinitesimal J decreases to only one of them as J increases to infinity. But also as J increases for fixed α the range of c for which unbound states exist increases from zero to infinity because we merely need $c^2 < J/\alpha^2$ to be able to solve the scattering problem.

The nth modified internal gravity wave satisfying (71) for small J can only cease to exist when $c \downarrow J_n^{\frac{1}{2}}/\alpha$ as $J \uparrow J_n$ for some value $J_n(\alpha)$. On this basis, we may take the leading term Φ_1 of an 'outer' expansion ϕ_0 of the solution of the Taylor–Goldstein equation (1) such that $\phi \to \phi_0$ as $J \uparrow J_n$ for fixed z, and therefore

$$\Phi_1'' + \left\{\frac{J_n}{(U - J_n^{\frac{1}{2}}/\alpha)^2} - \frac{U''}{(U - J_n^{\frac{1}{2}}/\alpha)} - \alpha^2\right\}\Phi_1 = 0. \tag{72}$$

This is the equation for the leading term in the outer expansion. We may separate the sinuous and varicose modes by taking the boundary conditions

$$\left.\begin{aligned} \phi_0 &= 1, \quad \phi_0' = 0 \quad n \text{ odd} \\ \phi_0 &= 0, \quad \phi_0' = 1 \quad n \text{ even} \end{aligned}\right\} \quad \text{at} \quad z = 0, \tag{73}$$

respectively. The outer solution is not uniformly valid at $z = \infty$, because if ϕ satisfies (1) and $\phi \to 0$ as $z \to \infty$ then

$$\phi \to \phi_i = A \exp\{-(\alpha^2 - J/c^2)^{\frac{1}{2}} z\} \tag{74}$$

as $z \to \infty$ for fixed J, for some constant $A(J)$. To match $\lim_{z\downarrow 0} \phi_i \approx \lim_{z\uparrow\infty} \phi_0$ as $J \uparrow J_n$, we find that we can match

$$\phi_0 \approx A(J) \quad \text{as} \quad z \to \infty, \quad J \uparrow J_n \tag{75}$$

to all orders in principle. Thus equation (72) with the two-point boundary conditions (73) and (75) pose an eigenvalue problem to determine J_n.

Now we can solve this problem with $\Phi_1 = 1$ if $J_n = \infty$, so we identify $J_1 = \infty$. As J decreases from infinity to $\alpha^2 U_{\max}^2 = \alpha^2$, we see that the coefficient of Φ_1 in equation (72) becomes larger so that $\Phi_1(z)$ oscillates more rapidly. Therefore $\alpha^2 < \ldots < J_3 < J_2 < \infty$, by the usual argument of the theory of ordinary differential equations.

With $J_1 = \infty$ the matching can be taken further. The expression in (74) is the exact inner solution apart from exponentially small terms, and we find that the outer solution ϕ_0 of the Taylor–Goldstein equation (1), such that $\phi \to \phi_0$ as $J \to J_1 = \infty$, is of the form

$$\phi_0 = \Phi_1 + J^{-\frac{1}{2}}\Phi_2 + J^{-1}\Phi_3 + \ldots$$

with

$$c_1 = J^{\frac{1}{2}}/\alpha + a_1 J^{-\frac{1}{2}} + a_2 J^{-1} + \ldots.$$

In order that the square-root in the exponent in (74) be positive we require that $a_1 J^{-\frac{1}{2}}$ is positive. The functions Φ_1, Φ_2 and Φ_3 are readily obtained seriatively. In the numerical results reported below the normalization used was such that

$$A = \exp\{\tfrac{1}{2}(\alpha^2 - J/c^2)^{\frac{1}{2}} \log 2\}.$$

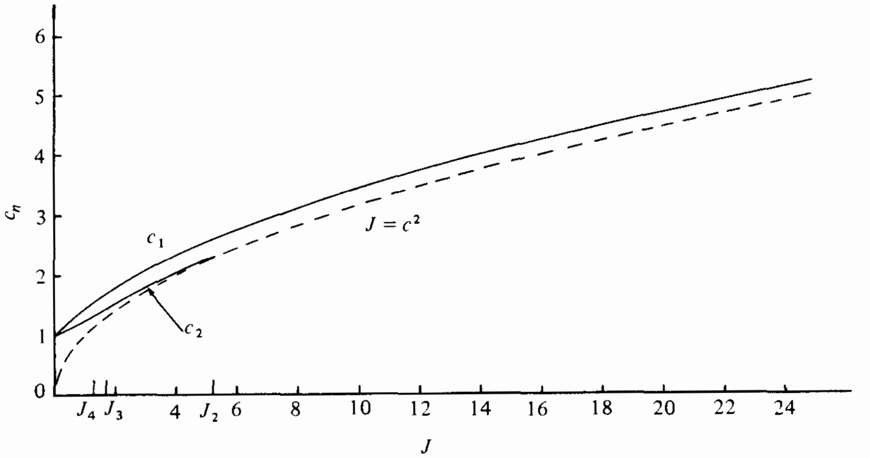

FIGURE 1. Bickley jet: $U = \operatorname{sech}^2 z$, $N^2 = 1$. c_n *vs.* J for the first two modes with $\alpha = 1$. The values of J_n ($n = 2, 3, 4$) are indicated and the curve $J = c^2$ is shown by a broken line.

After matching the outer solution to the suitably expanded and normalized inner solution, we find

$$\Phi_1 = 1, \quad \Phi_2 = \alpha^3 \log(\tfrac{1}{2}U) - \alpha U,$$

$$\Phi_3 = 11\alpha^4 U/6 - \alpha^4(\tfrac{1}{3} + U)\log(\tfrac{1}{2}U) + \tfrac{1}{2}\alpha^6[\{\log(\tfrac{1}{2}U)\}^2 - 4\log(\tfrac{1}{2}U) - (\log 2)^2]$$

and

$$a_1 = 2\alpha^3, \quad a_2 = -4\alpha^4(2\alpha^2 + \tfrac{1}{3}).$$

Consequently the eigenvalue relation is given by

$$c_1 = J^{\frac{1}{2}}/\alpha + 2\alpha^3 J^{-\frac{1}{2}} - 4\alpha^4(2\alpha^2 + \tfrac{1}{3})J^{-1} + o(J^{-1}), \quad \text{as} \quad J \to \infty, \tag{76}$$

where the positive square-root is taken.

The flow characterized by (70) has also been investigated by numerically integrating equation (1). We have used tanh z as the independent variable and, because of symmetry, confined attention to the interval (0, 1) by appropriately choosing the boundary condition at the origin. A Taylor expansion about $\tanh z = 1$ was used to provide the boundary conditions to start the integration, and by 'shooting' the two-point eigenvalue problem was solved. The integration incorporated an automatic change of step-length to achieve a specified uniform accuracy throughout. The method adopted was to use formula (71) to provide initial estimates for the eigenvalues for small J and then, helped by extrapolation, we proceeded to calculate the eigenvalues for progressively larger values of J.

Results were obtained with $\alpha = 1$ for the first four modes with

$$c_n > U_{\max} \quad (n = 1, 2, 3, 4)$$

and some of these are presented in figure 1. The outer problem defined by (72), (73) and (75) was also solved numerically and we were thus able to verify that $J_1 = \infty$ and that $J_2 = 5{\cdot}186$, $J_3 = 1{\cdot}680$, $J_4 = 1{\cdot}262$. The eigenvalues for large J were found to agree with (76). We have also found very good agreement between the analytical and numerical predictions of $\phi_0(0)$ for various large values of J. Other values of α were considered to test further the validity of (76).

For $c < U_{\min} = 0$, our numerical investigation suggests that there are no bound states. Further, if the method of matched asymptotic expansions is applied in strict analogy with equations (44)–(52) of Banks *et al.* (1976), it may at length be shown that the first approximation to the eigenvalue relation for the present problem gives no stable eigenvalue as $J \downarrow 0$, $c \uparrow 0$, thus supporting the numerical investigation. Additional evidence to this effect comes from a related problem considered in § 8.

7. Hyperbolic-tangent shear layer

For the basic flow

$$U = \tanh z \quad \text{and} \quad N^2 = 1 \quad \text{for} \quad -\infty < z < \infty, \tag{77}$$

some analytic results have been known a long time. Drazin (1958) showed that

$$J = \alpha^2(1-\alpha^2) \quad \text{and} \quad c = 0 \tag{78}$$

gives a neutral curve and Howard (1963) showed that there is instability for points just inside this curve in the first quadrant of the (J, α) plane. The asymptotic results (53)–(58) suggest that this is not the whole picture, however, so we integrated (1) numerically to obtain the stability characteristics and found that there is indeed more to be reported.

The numerical procedure we used is similar to that described in § 6, although the present problem is, of course, not symmetric. Further, z was used as the independent variable and the boundary conditions were usually imposed at $z = \pm 7$ to approximate infinity. However, to achieve sufficient accuracy for small values of α it was sometimes necessary (e.g. to verify the long-wave result (53)) to extend the range to $z = \pm 8$. Also we obtained some of the eigensolutions by posing and solving the scattering problem, and then finding those points in the (J, α) plane where the transmission and reflexion coefficients are singular, i.e. where there is resonance.

Our results are somewhat similar to those of Drazin & Davey (1977), who considered the stability of a shear layer in a compressible homogeneous inviscid fluid. Instead of the parameters α and J they have α and M, where M is the Mach number; M^2 is the analogue of J/α^2, as noted after (53). Also note that if $U(z)$ is an odd and $N^2(z)$ an even function then the existence of an eigensolution $\phi(\alpha, J, c, z)$ of (1), (2) implies the existence of another eigensolution $\phi^*(\alpha, J, -c^*, -z)$. Hence for each mode with $c_r > 0$ there exists a conjugate mode with the same values of α, J and c_i but with the opposite phase speed $-c_r$.

We present the results in graphical form, discussing the essential characteristics as necessary. In figures 2, 3 and 4 both components of the complex phase speed are plotted against the wavenumber α for 'typical' values of J/α^2. In these figures we have adopted the convention that when $c_r = 0$ the complex part of the phase speed is denoted by a broken line whereas when $c_r \neq 0$ it is denoted by a continuous line. These results may be better appreciated by reference to figure 5, where the contours of c_i are plotted in terms of α and J/α^2 for $c_i = 0{\cdot}01$ and $c_i = 0$. In finding these results we have been motivated by a desire to establish the main features and have consequently not tried to consider every detail. For example, the contour $c_i = 0$ in figure 5 is an *estimate*†

† We could have reached this conclusion more efficiently if we had followed the good advice of Banks *et al.* (1976, p. 164, para. 2) to divide out the singularity of equation (1) and solve the resultant Howard equation numerically.

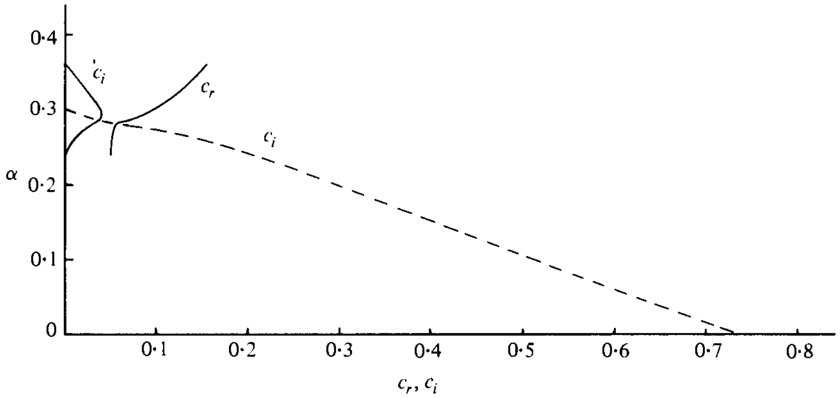

Figure 2. Shear layer: $U = \tanh z$, $N^2 = 1$. Variation of c_r and c_i with α for $J/\alpha^2 = 0{\cdot}91$. c_i is shown by a continuous line when $c_r \neq 0$, and by a broken line when $c_r = 0$.

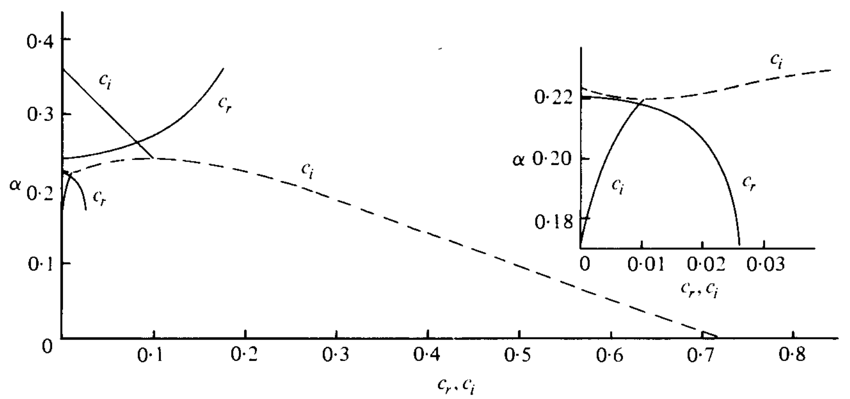

Figure 3. Caption as for figure 2 but with $J/\alpha^2 = 0{\cdot}95$.

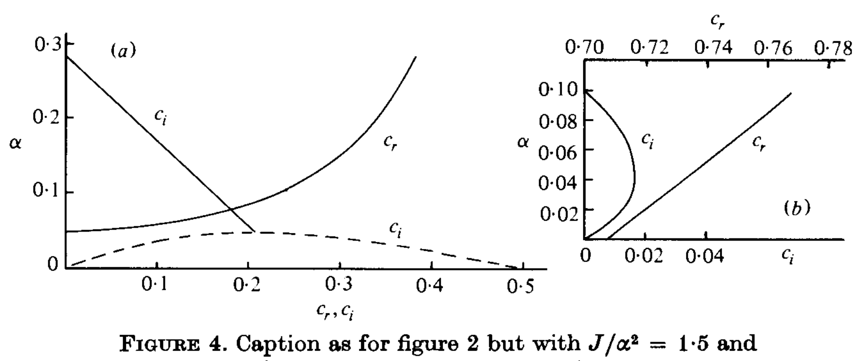

Figure 4. Caption as for figure 2 but with $J/\alpha^2 = 1{\cdot}5$ and $J/\alpha^2 = 3$ in (*a*) and (*b*) respectively.

based upon many of the numerical and asymptotic results presented in this paper and upon some numerical results not presented, notably a calculation of the contour $c_i = 0{\cdot}001$. In figure 5 we have also indicated some of the lines $c_r =$ constant, and, in particular, the contour $c_r = 0$ which depicts the bifurcation line outside which $c_r \neq 0$ and inside which $c_r = 0$. The nature of the bifurcations in figure 3 (which corresponds to there being two values of α for each value of $J/\alpha^2 < 1$ on the bifurcation line in figure 5) is clearly discernible. Figure 2, however, shows as clearly that there is no bifurcation

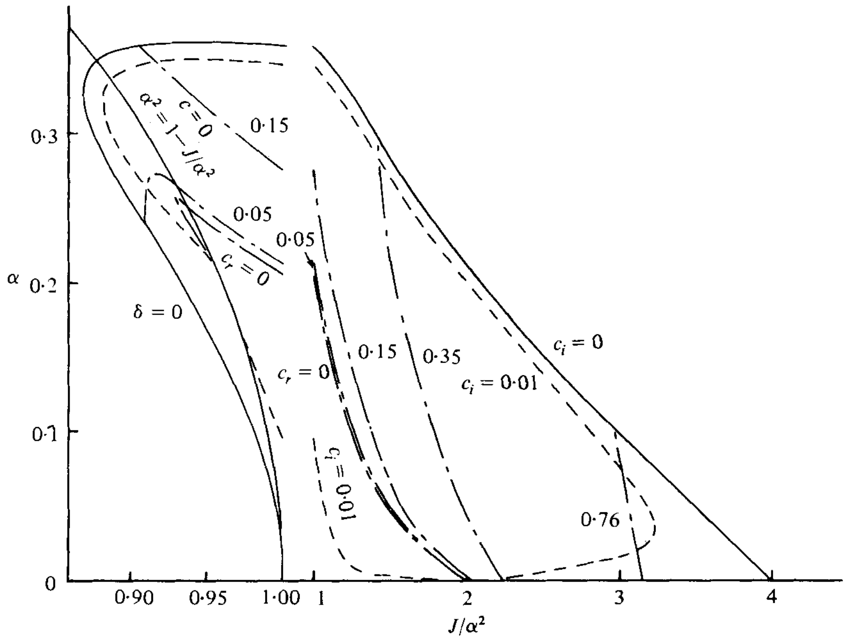

FIGURE 5. Shear layer: $U = \tanh z$, $N^2 = 1$. Contours of $c_i = 0{\cdot}01$ (broken) and $c_i = 0$ (continuous) in the $(J/\alpha^2, \alpha)$ plane. Lines of constant c_r (chain) are also shown, including the line of bifurcation $c_r = 0$.

at $J/\alpha^2 = 0{\cdot}91$, even though the slope of the c_r-curve changes rapidly. We have not tried to find very accurately the least value of J/α^2 at which bifurcation occurs nor indeed the shape of the bifurcation curve near this minimum value; the value is approximately 0·93. Also in figure 6 we have plotted c_r as a function of J/α^2 for $c_i = 0{\cdot}01$ and 0. The whole curve for $c_i = 0$ is an estimate on the same basis as the marginal curve in figure 5. The dotted curve in figure 6 corresponding to $c_i = 0$ between (2, 0) and (4, 1) represents the first-order term in the first equation of (56), while the dotted curve corresponding to $c_i = 0{\cdot}01$ between (2, 0) and (2·95, 0·7) is based upon both equations in (56). We note the satisfactory join of analytical and numerical results.

We chose $J/\alpha^2 = 0{\cdot}91$ (figure 2) and 0·95 (figure 3) as values typical of bifurcation being absent and present respectively. The value $J/\alpha^2 = 1{\cdot}5$ is typical of the interval (1, 2) and $J/\alpha^2 = 3$ of the interval (2, 4) (figure 4). We have compared many of these numerical results with the asymptotic results (53) and (55)–(58), and the agreement is satisfactory.

For more convenient application of the theory we present the marginal curve in the (J/α) plane in figure 7. This emphasizes that the picture is more complicated than the previously accepted one of a single unstable mode within the curve (78), but, of course, the critical value of the Richardson number is still $\frac{1}{4}$.

Einaudi & Lalas (1976) considered the same basic flow (77) but with two rigid horizontal walls, and found an infinity of unstable modes which vanish to leave only Drazin's mode (78) in the limit as the walls recede to infinity. This suggests an interpretation of the extra modes we have just described. They are plausibly due to the

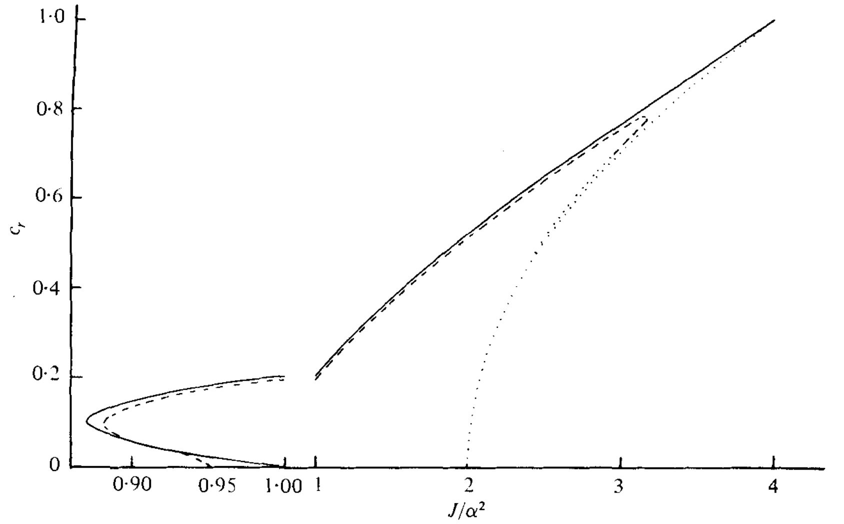

FIGURE 6. Shear layer: $U = \tanh z$, $N^2 = 1$. c_r *vs.* J/α^2 for $c_i = 0{\cdot}01$ (broken) and $c_i = 0$ (continuous). The parts of the contours shown dotted are taken from the analytical results.

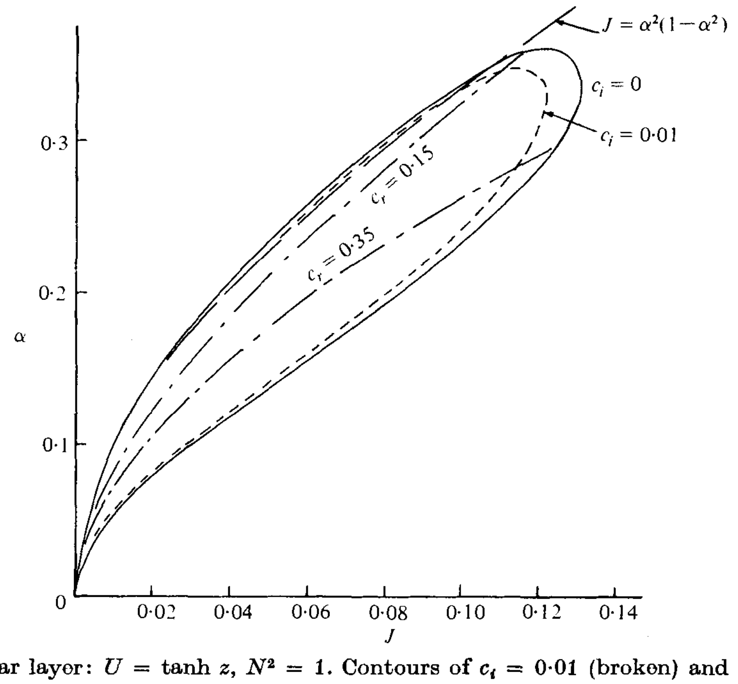

FIGURE 7. Shear layer: $U = \tanh z$, $N^2 = 1$. Contours of $c_i = 0{\cdot}01$ (broken) and $c_i = 0$ (continuous) in the (J, α) plane. Lines of constant c_r (chain) are also shown.

propagation at infinity possible in an unbounded flow, whereas those extra modes found by Einaudi & Lalas are due to the trapping of waves by reflexion between the walls. This explains the qualitative difference between the results in the absence and presence of walls, however distant the walls may be.

We are unable to find any bound states (with $c > U_{max} = 1$ or $c < U_{min} = -1$) for the flow (77), either numerically, or by applying the method of matched asymptotic expansions (as in the previous section) as either $J \downarrow 0$, $c \downarrow 1$ or $J \downarrow 0$, $c \uparrow -1$.

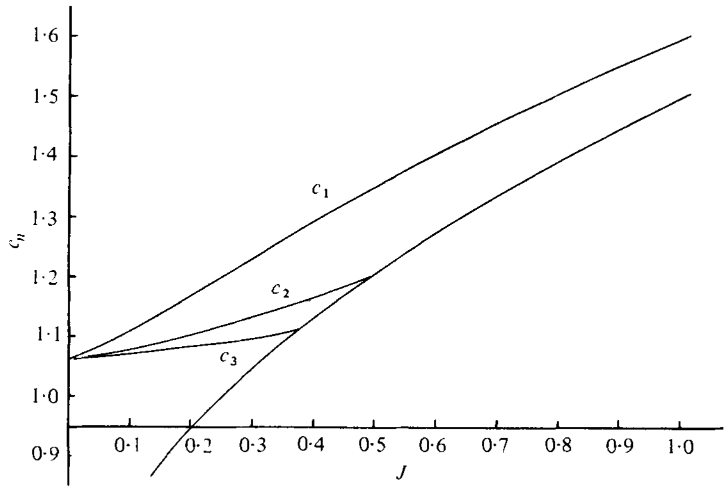

FIGURE 8. $U = L\,\mathrm{sech}^2\,z + M\tanh z$, $N^2 = 1$. c_n *vs.* J for the first three modes with $L = 1$, $M = \frac{1}{2}$ and $\alpha = 1$. The curve $c = M + J^{\frac{1}{2}}$ is also shown; for fixed J the difference between this and the c_1 curve is less than 0·1 where $J = 1$ and decreases to zero as J increases to J_1.

The apparent absence of such modes led us to consider the 'hybrid' flow

$$U = L\,\mathrm{sech}^2\,z + M\tanh z \quad \text{and} \quad N^2 = 1 \quad \text{for} \quad -\infty < z < \infty. \tag{79}$$

As L/M varies this indicates how the bound states cease to exist. We further choose $M > 0$ without loss of generality. In order to have solutions ϕ decaying at infinity we require $(c - M)^2 > J/\alpha^2$. The flow (79) gives $U' = 0$ at $z = \pm\infty$ and also where $\tanh z = M/2L$ if $M < 2L$. The latter gives a maximum value of U, $U_{\max} = L + M^2/4L$ if $L > 0$; then, as M/L increases from 0 to 2 so the position at which U attains its maximum varies from $z = 0$ to $z = \infty$. For $J \ll 1$ stable eigenvalues corresponding to $U_{\max}$ (i.e. springing from $c = L + M^2/4L$, $J = 0$) are expected as in the work of Banks *et al.* (1976), provided that $L + M^2/4L > M$. As M/L increases to 2 these modes are confined to an ever smaller region of the (J, c) plane and eventually disappear completely. A numerical investigation was made using as typical the particular values $L = 1$, $M = \frac{1}{2}$ with $\alpha^2 = 1$. For small values of J agreement with Banks *et al.* (1976, equation (43)) is good: for example, at $J = 0{\cdot}1$ the computed values of c for the first three modes ($n = 1, 2, 3$) are 1·1080 (1·1004), 1·0791 (1·0767) and 1·0711 (1·0701), where the corresponding asymptotic results are shown in parentheses. For this choice of L, M and α^2 a finite value for J_1 ($= 14{\cdot}94$) was found, while approximate values for J_2 and J_3 were 0·51 and 0·38 respectively. The variation of c_n ($n = 1, 2, 3$) with J is shown in figure 8. The outer problem was then solved numerically, exactly as in § 6, in order to determine J_1 for a range of values of M/L where L was taken to be unity without loss of generality. As expected, J_1 was found to decrease from infinity as M/L increased from zero.

8. An exact solution, after Miles

For the basic flow

$$U = 1 - e^{-z} \quad \text{and} \quad N^2 = 1 \quad \text{for} \quad 0 \leqslant z < \infty, \tag{80}$$

there is an exact analytic solution. This profile was examined to confirm the absence of the modified internal gravity waves for both jet and shear layer profiles. But also exact solutions are of intrinsic interest and are desirable to record for use as examples.

Following Miles (1967), let $\phi = e^{-\alpha z} f(w)$, where $w = e^{-z}/(1-c)$. Then we find

$$f(w) = w^{\lambda_+ - \alpha}(1-w)^{\frac{1}{2}(1+\sigma)} F(a, b;\ 1+2\lambda_+;\ w), \tag{81}$$

where $\sigma = \{1 - 4J/(1-c)^2\}^{\frac{1}{2}}$, $\operatorname{Re}\sigma \geqslant 0$, $\lambda_+ = \{\alpha^2 - J/(1-c)^2\}^{\frac{1}{2}}$, $\operatorname{Re}\lambda_+ \geqslant 0$,

and $$a, b = \tfrac{1}{2}(1+\sigma) + \lambda_+ \pm (1+\alpha^2)^{\frac{1}{2}}.$$

This solution (81) satisfies (1) and $\phi \to 0$ as $z \to \infty$; the remaining condition $\phi = 0$ at $z = 0$ gives the eigenvalue relation

$$F(a, b;\ 1+2\lambda_+;\ 1/(1-c)) = 0. \tag{82}$$

Thus we require the distribution of the zeros of the hypergeometric function F. We use the results of Klein relating to the real zeros (cf. Van Vleck 1902) in the case of real canonical parameters of the hypergeometric function, i.e. λ_+ and σ both real. We find that if $0 < J/(1-c)^2 < \min\{g_0(\alpha), \frac{1}{4}, \alpha^2\}$, where

$$g_0(\alpha) = \{(1+\alpha^2)^{\frac{1}{2}} - 1\}/\{\tfrac{5}{2} - 2(1+\alpha^2)^{\frac{1}{2}} + 2\alpha^2\},$$

then there is no eigenvalue c in $(-\infty, 0]$, while if $g_0(\alpha) < J/(1-c)^2 < \min(\frac{1}{4}, \alpha^2)$ then there is just one eigenvalue in $(-\infty, 0]$. Further, if $J/(1-c)^2 < \min(\frac{1}{4}, \alpha^2)$ then there is no eigenvalue in $[1, \infty)$.

The eigenvalue relation (82) may be rewritten in the form

$$F(a, a-2\lambda_+;\ 1+2(1+\alpha^2)^{\frac{1}{2}};\ 1-c) = 0. \tag{83}$$

Applying the results of Klein to this form we find that for $J/(1-c)^2 < \min(\frac{1}{4}, \alpha^2)$ there are no zeros of (83) with $1-c$ in $(0, 1)$ and thus no stable eigenvalue c in $(0, 1)$.

The single bound state that has been found above arises as σ decreases through $\sigma_0(\alpha) = 2(1+\alpha^2)^{\frac{1}{2}} - 2\lambda_+ - 1$, i.e. $J/(1-c)^2$ increases through $g_0(\alpha)$, and is such that $c \sim -\{\frac{1}{2}(\sigma_0 - \sigma) B(1+2\lambda_+, \sigma_0)\}^{1/\sigma_0}$ as $(1-4J)^{\frac{1}{2}} \to \sigma_0(\alpha)$, where B is the beta function.

Next we consider $\frac{1}{4} < J/(1-c)^2 < \alpha^2$ so that σ is pure imaginary. Following Miles, we find the limiting eigenvalue relation

$$(4J-1)^{\frac{1}{2}} \cot[(4J-1)^{\frac{1}{2}} \log\{-c/(1-c)\}] = \{2\psi(1) - \psi(a_0) - \psi(b_0)\}^{-1} \quad \text{as} \quad J \downarrow \tfrac{1}{4}, \quad c \uparrow 0, \tag{84}$$

where $a_0, b_0 = \frac{1}{2} + \lambda_+ \pm (1+\alpha^2)^{\frac{1}{2}}$ and ψ is the psi or digamma function. The left-hand side of (84) takes any value in $(-\infty, \infty)$ an infinite number of times as $c \downarrow 0$ with $J > \frac{1}{4}$. It follows that (84) has an infinite number of roots with a limit point at $c = 0$ for $J > \frac{1}{4}$ and any $\alpha > \frac{1}{2}$, this last condition being necessary in order to satisfy the boundary condition at infinity.

In the limit as $\alpha \to \infty$ we find that the eigenvalues are given by the roots of $K_{\frac{1}{2}\sigma}\{-\alpha c/(1-c)\} = 0$, where $K_{\frac{1}{2}\sigma}$ is a modified Bessel function of the second kind.

For the limit as $J \to \infty$ there is no obvious analogue of Miles' asymptotic result for the eigenvalues: the restriction $\alpha^2 > J/(1-c)^2$ ensures that as $J \to \infty$ the canonical parameters of the hypergeometric function in the eigenvalue relation remain finite in general. However, if α is large so that $c = J^{\frac{1}{2}}/\alpha + O(1)$ and $\alpha^2 - J/(1-c)^2 \downarrow 0$ as $J \to \infty$, then the hypergeometric series is approximated by $J_0\{2\alpha(2w)^{\frac{1}{2}}\}$ and the eigenvalue relation is then satisfied by $(1-c)^{-1} = j_{0,n}^2/8\alpha^2$, where $j_{0,n}$ is the nth positive zero of the Bessel function J_0. This gives

$$c \sim -8\alpha^2/j_{0,n}^2 \quad \text{as} \quad J \sim 64\alpha^6/j_{0,n}^4 \quad \text{and} \quad \alpha \to \infty.$$

The basic flow (80) may also be used to form a jet: we may define

$$U = \begin{cases} 1-e^{z} & \text{for} \quad z > 0, \\ 1-e^{-z} & \text{for} \quad z < 0. \end{cases} \tag{85}$$

(A superposition of a uniform flow field followed by a change of sense produces a standardized jet with $U_{\pm\infty} = 0$.) The exact solution presented in this section, with $\phi(-z) = -\phi(z)$ is then also a solution for the odd (varicose) modes in the jet (85), because the condition $\phi = 0$ at $z = 0$ has been satisfied, and continuity of $\phi/(U-c)$ and of $(U-c)\,\phi' - U'\phi$ thereby also hold at $z = 0$. But the analysis indicates that there are no stable modes for which $c \downarrow 1$, so that there are no stable odd modes for the jet (85) with $c \downarrow 1$; this corresponds to $c \uparrow 0$ for the standardized jets of previous sections and thus agrees with our findings for these jets.

9. Conclusions

In the introduction we classified the normal modes in five types of bound states and three types of unbound states. In the rest of the paper we have presented a mosaic of numerical and asymptotic results to build up the overall picture of these linear modes. We depicted the complementary nature of the spectra of the bound states which occur when λ_+^2 and λ_-^2 are positive and the unbound states which occur otherwise. We have treated the bound states similarly before (Banks *et al.* 1976), but our new asymptotic and numerical results for shear layers reveal new modes of instability. On their stability boundaries these new modes become waves propagating at infinity, but seem not to affect the criterion of stability of the basic shear layer.

Some of our results on unbound states, developing those of Booker & Bretherton (1967) and other authors, distinguish the character of the scattering sharply according to whether there is a critical layer or not, i.e. according to whether c lies in the range of the basic velocity or not. Others of our results relate the over-reflexion of the unbound states to the stability boundary of the bound states. We have seen that a propagating marginally stable mode may also be regarded as a scattered wave with infinite transmission and reflexion coefficients. In other words, infinite over-reflexion is a resonance. The presence of contiguous unstable modes means that this resonance occurs only when the basic flow is unstable. The growth rates of these unstable modes are, however, usually slow, so that waves with a reflexion coefficient greater than one may build up before the basic flow breaks down into turbulence.

For the basic vortex sheet (45), it may appear that there is an infinite reflexion or transmission coefficient for a wave to which the flow is stable. This is true in one sense, but we have shown it is unrealistic in the sense that for any smoothly varying profile, however closely it may approximate the discontinuous vortex sheet, there are unstable modes contiguous to the resonance. Again, the growth rates of these unstable modes vanish in the limit as the smooth profiles tend to the discontinuous one. Thus the use of a discontinuous profile, which has a degenerate critical layer at its discontinuity, has a restricted value as an approximation to a smoothly varying profile.

To assess properly the significance of rapid variation of the profile and of slow growth rates, nonlinearity also should be considered. Grimshaw (1976*b*) and McIntyre & Weissman (1978) have already considered weak nonlinearity and slow growth rates for discontinuous profiles and basic stratified flows, but it now appears that some of their results may be qualitatively different for rapidly varying shear layers.

Most of the ideas developed in this paper can be applied to many similar problems, e.g. when the basic flow is of a compressible fluid, is in a rotating frame, or is of an electrically conducting fluid in a magnetic field. Indeed, over-reflexion was recognized by Lees & Lin (1946) and by Miles (1957) for the problem of parallel flow of an inviscid compressible fluid. The importance of over-reflexion, of extra modes propagating at infinity, and of critical layers in subsequent work is indicated well by Acheson (1976). Our conclusions in part concur with Acheson's, but in part are at variance with them (Acheson 1976, p. 434). His work is based largely upon the use of discontinuous profiles, though we have shown an essential difference between them and rapidly but smoothly varying profiles. Our results suggest that resonant over-reflexion coexists with instability or at any rate is a mode contiguous to unstable ones for any smoothly varying profile. Whether the instability would break up the basic flow before over-reflexion could occur in practice would depend upon the relative importance of a number of small departures of the real flow from the idealized model. Acheson noted that the instability might be so weak as to permit over-reflexion for some initial period, and also recognized that nonlinearity and diffusivity, effects which we have not considered, may be important in practice. Further, we note that real flows are neither steady nor parallel and horizontal, and that a sinusoidal disturbance is not produced instantly but rather is one component in the representation of a disturbance as it evolves in time. All these factors must be borne in mind when interpreting the idealized theory we have treated here.

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